

Non-universal exponents in interface growth

T. J. Newman¹ and Michael R. Swift²

¹*Department of Theoretical Physics, University of Manchester,
Manchester, M13 9PL, UK*

²*International School for Advanced Studies,
via Beirut 2-4 and Sezione INFN di Trieste, I-34013*

Abstract

We report on an extensive numerical investigation of the Kardar-Parisi-Zhang equation describing non-equilibrium interfaces. Attention is paid to the dependence of the growth exponent β on the details of the distribution of noise $p(\xi)$. All distributions considered are delta-correlated in space and time, and have finite cumulants. We find that β becomes progressively more sensitive to details of the distribution with increasing dimensionality. We discuss the implications of these results for the universality hypothesis.

PACS numbers: 05.40.+j, 68.35.R

The Kardar-Parsis-Zhang (KPZ) equation [1–3], although originally introduced as a model of non-equilibrium interface growth, has acquired a broader significance over the past decade as one of the simpler examples of a strong-coupling system. To this date there exists no systematic analytic procedure for determining the properties of the model for large values of the non-linear coupling. Additional interest in the model stems from its intimate mathematical relation to two other important systems, namely the noisy Burgers equation [4], and the equilibrium properties of a directed polymer in a random medium (DPRM) [3].

Most work on this problem to date has concentrated on determining the properties of the system (e.g. values of dynamic exponents) in the strong-coupling phase. In pursuit of this goal, the ‘hypothesis of universality’ (HOU) has been generally adopted; namely that details of the model representation of the KPZ equation should not affect universal quantities such as exponent values. In this Letter we report on an extensive numerical investigation of the KPZ equation which shows that exponents are sensitive to the precise form of the noise distribution, this sensitivity becoming extreme in higher dimensions. We regard our results as being *strongly suggestive* of a breakdown of universality in the KPZ equation, but of course, they cannot constitute a proof of this assertion. To set the scene for what is to come we devote one paragraph below to the general theoretical framework of the KPZ equation, and a further paragraph to the current state of knowledge.

Denoting the interface profile by $h(\mathbf{x}, t)$ (which is defined perpendicular to the d -dimensional substrate), the KPZ equation has the form

$$\partial_t h = \nu \nabla^2 h + \lambda (\nabla h)^2 + \xi , \quad (1)$$

where $\xi(\mathbf{x}, t)$ is a stochastic source, generally taken to be delta-correlated in space and time: $P[\xi] = \prod_{\mathbf{x}} \prod_t p(\xi(\mathbf{x}, t))$. The canonical choice for the distribution p is a gaussian, and the HOU is invoked to argue that any other choice (so long as it has finite cumulants) will lead to the same large-scale behavior. The prime goal is the computation of the exponents which characterize the evolution of fluctuations, along with the value of the upper critical dimension d_u , above which one expects the exponents to saturate at their mean-field values.

In the language of the KPZ equation, there are three exponents of immediate interest: Starting from a flat interface, the mean square fluctuations grow as $W^2 \equiv \langle h^2 \rangle_c \sim t^{2\beta}$, but will saturate for large times in a finite system of linear dimension L – the saturated value is expected to scale as $W \sim L^\chi$. The fluctuations over the entire temporal regime may be conveniently described by the two-point correlation function $C(\mathbf{r}, t) \equiv \langle (h(\mathbf{r}, t) - h(\mathbf{0}, t))^2 \rangle \sim r^{2\chi} f(r^z/t)$. The exponent z is often known as the dynamic exponent and gives the fundamental scaling between length and time. The three exponents may be reduced to one independent exponent by two scaling laws: $z\beta = \chi$ and $z + \chi = 2$. The first law comes from the scaling behavior of C as $r \rightarrow \infty$, and the second is a consequence of invariance of the equation under an infinitesimal tilt transformation. It is also worth mentioning that the non-linear transformation $w = \exp(\lambda h/\nu)$ produces the linear equation

$$\partial_t w = \nu \nabla^2 w + (\lambda/\nu) w \xi, \quad (2)$$

which corresponds to the equation for a $(d + 1)$ -dimensional directed polymer in a random potential $-\lambda\xi$.

Given the extreme difficulty of any analytic progress, most investigations of the KPZ equation have been numerically based. These investigations fall into two categories. The majority of numerical work consists in simulating microscopic models which under the HOU are assumed to have the same large-scale behavior as the KPZ equation. Such models include Eden growth [5], polynuclear growth models [6], the restricted solid-on-solid (RSOS) model [7,8], and its close relative, the hypercube stacking model [9]. The second numerical approach is that of direct integration of the equation itself [10,11]. This entails some subtleties of discretization which have only recently come to light [12]. However, there has been reasonable agreement between these various approaches as regards the values of exponents, at least in $d = 1 + 1$ (where there exists an exact analytic result of $z = 3/2$ [13]) and $d = 2 + 1$. In higher dimensions, numerical work becomes more difficult due to the smaller linear dimension of the systems (and the consequent early onset of finite size effects), but still there has been general agreement that although exponents may not be so precisely de-

terminated, there is no sign of a crossover to mean field values ($z = 2$) for dimensions up to $d = 7 + 1$ [8]. An interesting analytic method which may be applied to the strong coupling regime is mode-coupling theory [14] (although it is based on an *ad hoc* neglect of vertex renormalization) which seems to support $d_u = 4$. Also there has been recent work [15] based on short-distance expansion techniques in the renormalization group (RG), which indicates the bound $d_u \leq 4$, whereas a mapping to directed percolation [16] suggests that $d_u \leq 5$.

Our original motivation for this numerical study was to integrate a recently proposed discrete equation [12], which was shown to correctly capture the strong-coupling behavior of the KPZ equation on a lattice. Only during the course of our work did we discover the noise sensitivity of the exponents, which is the focus of this Letter. Nevertheless, before presenting our results, it is useful to briefly describe this improved algorithm, and also to exhibit its close relation to the algorithm for zero temperature DPRM [3], which has also been numerically studied in the past [17].

The key point concerning the discretization of the KPZ equation, is that one is only guaranteed to capture the strong-coupling physics by discretizing the directed polymer equation (2), and constructing the discrete KPZ equation by the inverse transform $h_i = (\nu/\lambda) \ln(w_i)$. The time discretization necessarily introduces a two-stage process – i) pumping with the noise, and ii) relaxing with the deterministic part of the equation. Explicitly one has the discrete KPZ equation in the form

$$\begin{aligned} \tilde{h}_i(t) &= h_i(t) + \Delta^{1/2} \xi_i(t) \\ h_i(t + \Delta) &= \tilde{h}_i(t) + (\nu/\lambda) \ln \left\{ 1 + (\Delta\nu/a^2) \sum_{j \text{ nn } i} \left[e^{\lambda(\tilde{h}_j - \tilde{h}_i)/\nu} - 1 \right] \right\} , \end{aligned} \quad (3)$$

where Δ and a are the grid scales for time and space respectively. Taking the strong-coupling limit $\lambda \rightarrow \infty$ allows one to write the relaxation stage of the above equation in the much simpler form

$$h_i(t + \Delta) = \max_{j \text{ nn } i} (\tilde{h}_i, \{\tilde{h}_j\}) . \quad (4)$$

This corresponds exactly to the zero-temperature DPRM algorithm but written here in

terms of the field h_i rather than w_i . Note that there are no adjustable parameters in the strong-coupling algorithm, except for the functional form of the noise distribution $p(\xi)$ (since the grid scales, and the noise strength may be scaled away).

We have implemented the above algorithm in dimensions $d + 1$ with $d = 2, 3, 4$, with a flat surface ($h_i(0) = 0$) as the initial configuration.. The only adjustable ‘parameter’ in our simulations has been the function $p(\xi)$. This function was taken to be either gaussian $p_g(\xi) \sim \exp(-\xi^2/2)$, or of the form $p_\alpha(\xi) \sim (\sigma - |\xi|)^\alpha$, $-\sigma \leq \xi \leq \sigma$ (where σ is adjusted to maintain unit variance for each choice of α). On varying α the distribution p_α interpolates through the forms: bimodal ($\alpha \searrow -1$), top-hat ($\alpha = 0$), triangular ($\alpha = 1$), and finally distributions with very rapidly vanishing tails as $\alpha \rightarrow \infty$. Note that all distributions are strongly localized and have an infinite set of finite cumulants. [They are thus distinct from distributions with *power-law tails*, which are known to change the exponent values [3].]

Our simulations are performed on lattices of size L^d , with averaging over N samples. The simulations were of the size ($L = 2048, N = 8$) for $d = 2$, ($L = 200, N = 8$) for $d = 3$, and ($L = 60, N = 8$) for $d = 4$. Although these simulations are of the largest scale possible within our resources (a cluster of DEC alpha workstations), we are aware that specialist computational groups can improve on the precision of our results. However, the qualitative features of interest in this Letter are convincingly clear from our simulations.

The interface width W is plotted as a function of time in Figs.1, 2 and 3, for dimensions $d = 2, 3$ and 4 respectively. As mentioned earlier, this quantity is expected to grow as $W \sim t^\beta$ (so long as the dynamic length scale is much less than L). It is clear that there is a dependence of β on the value of α , this dependence becoming progressively stronger in higher dimensions. (This is the reason for not presenting data in dimension (1+1), for which all distributions yield a value of β close to the exact result of 1/3 [13]). The measured values of β are presented in Table 1. An important point to make is that in all dimensions, our results are in agreement with the consensus of exponent values in the literature [8,9,17], *if* we use the gaussian noise $p_g(\xi)$. However, the values of β drops smoothly as the parameter α is decreased.

Naturally one may attempt to interpret these results in terms of a temporal crossover. Given the straightness of the curves, such a crossover would be exceptionally slow. We have checked for the presence of this phenomenon by trying to fit the data with the generic form $W^2 = At^{2\beta} + Bt^{2\gamma}$ (which implicitly includes the popular fitting Ansatz $\gamma = 0$ used by previous groups [9,17]). The best fits correspond to values of β close to the value one would obtain by the simpler fit $W \sim t^\beta$, with the ‘correction to scaling’ exponent γ taking a value of approximately $\beta/2$. This indicates that the curves show no sign of crossover, as they have a clearly dominant power-law form, with modest corrections. The quoted errors in β (shown in Table 1) are estimated from the range of β with which one may obtain an acceptable fit using the above fitting Ansatz.

To enable a check of our results, we have concentrated on the value $\alpha = 1/2$ in $d = 3$ for which we averaged over $N = 64$ samples of size $L = 180$. In this case, the data is of a good enough quality to measure a running value of the growth exponent $\beta_{\text{eff}}(t) = d \ln(W)/d \ln(t)$ [17]. The quality of the data obtained for the value $\alpha = -1/2$ (with $N = 8$ samples) is also sufficiently good to allow this measurement. The running exponents $\beta_{\text{eff}}(t)$ for $\alpha = 1/2$ and $\alpha = -1/2$, plotted against $t^{-\beta}$ (which magnifies potential systematic deviations from a simple power law), are shown in the insets of Figs. 4 and 5 respectively; with the previously measured values of β used in each case. There is no sign of any asymptotic deviation away from these values of $\beta = 0.14$ and $\beta = 0.05$.

Another independent check was made for these two values of α , by studying the variation of the saturated (or steady-state) width W_{SS} , as a function of the system size. On scaling grounds one expects $W_{SS} \sim L^\chi$, with $\chi = 2\beta/(1 + \beta)$, where we have assumed the exponent relations $z\beta = \chi$ and $z + \chi = 2$. In Figs.4 and 5, we plot W_{SS} against L for $\alpha = 1/2$ and $\alpha = -1/2$ respectively. One observes that there is more than a decade of clean scaling in each case, with fitted values $\chi = 0.24$ and $\chi = 0.08$ respectively, which are consistent (within the stated errors) with the previously measured values of β . On the grounds of this numerical work we are led to the statement that the growth exponent β is non-universal with respect to changes in the noise distribution $p(\xi)$. In the remainder of the paper we

present a brief discussion of the robustness and physical implications of this claim.

It is possible to gain information about the exponents from physical quantities other than W . As an example, one may measure the two-point correlator $C(\mathbf{r}, t)$ (or its Fourier transform, the structure factor) and attempt to collapse the functions using dynamical scaling. We have been able to produce tolerably good data collapse for $C(\mathbf{r}, t)$ in all cases, yielding values of the exponents within the errors of those listed in Table 1. However, in our opinion the measurement of exponents from data collapse is less reliable than direct measurements of W due to the difficulties of including corrections to scaling in a systematic fashion.

As mentioned above, one may try to interpret these results in terms of slow temporal crossover. Our analysis indicates that strong crossover effects are absent from the data, since we are able to measure reasonably precise values of the exponent β along with its ‘correction to scaling exponent’ γ . However, one can not rule out the possibility that one has simply failed to reach the ‘true’ asymptotic regime (AR), and that one is measuring some transient scaling regime. This introduces the question of how one should empirically define the AR. For our purposes we have used the standard working definition that in the AR one observes clean power-law behavior of the quantities of interest (in our case the interface width W), and that the correlation length in the system is large compared to the lattice scale. Both criteria have been met in our simulations. A further criterion which one may invoke [18] is that the interface width itself must be much greater than the effective lattice spacing in the growth direction (here set to $O(1)$ by the noise variance being normalized to unity.) This criterion is not satisfied in our simulations for $d = 4$, or for very small α , for the simple reason that if the interface fluctuations grow very slowly with time (meaning β is small) then one will never achieve $W \gg 1$ on observable time-scales, even though the correlation length of the system is large (since z is generally close to 2). There can be no definitive answer to the question of whether a given simulation has reached the AR, although we would encourage specialist computational groups to improve on our time scales and system sizes in order to shed more light on this question.

As a final comment on the simulations, it is possible that these results are a consequence of working at the strong-coupling limit ($\lambda \rightarrow \infty$). An example is known in the directed polymer literature [19] of exponents dependent on the noise distribution at zero temperature – for the case of perfectly correlated disorder in the longitudinal direction. The reasons for this dependency are physically clear and may be illustrated by a simple Flory argument. However, it is doubtful whether a similar effect is relevant to our simulations, in which the noise is delta-correlated in time.

If the HOU is false for the KPZ equation, it is crucial to trace the underlying physical reasons for this. The most fragile aspect of KPZ physics is the role of the microscopic cut-off. For instance, the fact that the naive (but canonical) discretization fails to retain the continuum physics [12] gives one real cause for concern. It may be that the lattice scale is always relevant to the large-distance scaling of the interface, which would then give scope for non-universal features, such as those seen in the present work. A similar example of non-standard scaling is known from controlled calculations on the deterministic Burgers problem [20].

In this paper we have given convincing numerical evidence that the KPZ growth exponent β is strongly sensitive to certain details of the noise distribution (here characterized by the parameter α). It is important to understand whether the results presented here are really the hallmark of a breakdown of the HOU for the KPZ equation, or whether there exist (exponentially) long crossover scales deep within the strong-coupling phase itself. Whichever is the case, we believe that these results are indicative of unexpected and interesting new physics in the KPZ problem.

The authors are grateful to Alan Bray, Joachim Krug, Amos Maritan and Michael Moore for useful conversations. TJN acknowledges financial support from the Engineering and Physical Sciences Research Council.

REFERENCES

- [1] M. Kardar, G. Parisi and Y.-C. Zhang, Phys. Rev. Lett. **56**, 889 (1986).
- [2] J. Krug and H. Spohn, in *Solids Far From Equilibrium: Growth, Morphology and Defects*, ed. C. Godrèche (Cambridge, Cambridge University Press, 1991).
- [3] T. Halpin-Healy and Y.-C. Zhang, Phys. Rep. **254**, 215 (1995).
- [4] J. M. Burgers, *The Non-linear Diffusion Equation* (Reidel, Boston, 1974).
- [5] M. Eden, in *Symposium on Information Theory in Biology*, ed. H. P. Yockey (Pergamon Press, New York, 1958).
- [6] F. C. Frank, J. Cryst. Growth **22**, 233 (1974); J. Kertész and D. E. Wolf, Phys. Rev. Lett. **62**, 2571 (1989).
- [7] J. M. Kim and J. M. Kosterlitz, Phys. Rev. Lett. **62**, 2289 (1989).
- [8] T. Ala-Nissila *et al.*, J. Stat. Phys. **72**, 207 (1993).
- [9] B. M. Forrest and L.-H. Tang, Phys. Rev. Lett. **64**, 1405 (1990); L.-H. Tang, B. M. Forrest and D. E. Wolf, Phys. Rev. A **45**, 7162 (1992).
- [10] J. G. Amar and F. Family, Phys. Rev. A **41**, 3399 (1989).
- [11] K. Moser, J. Kertész and D. E. Wolf, Physics A **178**, 215 (1991); K. Moser and D. E. Wolf, J. Phys. A **27**, 4049 (1994).
- [12] T. J. Newman and A. J. Bray, J. Phys. A **29**, 7917 (1996).
- [13] D. A. Huse, C. L. Henley and D. S. Fisher, Phys. Rev. Lett. **55**, 2924 (1985). [Note this result is proven only for gaussian noise.]
- [14] M. A. Moore, T. Blum, J. P. Doherty, M. Marsili, J.-P. Bouchaud, and P. Claudin, Phys. Rev. Lett. **74**, 4257 (1995).
- [15] M. Lässig and H. Kinzelbach, Phys. Rev. Lett. **78**, 903 (1997).

- [16] M. Cieplak, A. Maritan and J. R. Banavar, Phys. Rev. Lett. **76**, 3754 (1996).
- [17] J. M. Kim, A. J. Bray and M. A. Moore, Phys. Rev. A **44**, 2345 (1991).
- [18] J. Krug, private communication.
- [19] J. Krug and T. Halpin-Healy, J. Phys. I (France) **3**, 2179 (1993).
- [20] S. E. Esipov and T. J. Newman, Phys. Rev. E **48**, 1046 (1993), T. J. Newman, Phys. Rev. E. (1997) to appear.

List of figure captions

Fig. 1: Interface width W^2 versus t in dimension $2 + 1$. The upper curve corresponds to p_g and the lower to p_α with $\alpha = 0$. The straight lines are fitted with values of β given in Table 1.

Fig. 2: Interface width W^2 versus t in dimension $3 + 1$. The uppermost curve corresponds to p_g and the lower curves to p_α with $\alpha = 2, 1, 1/2, 0$ and $-1/2$ in descending order.

Fig. 3: Interface width W^2 versus t in dimension $4 + 1$. The uppermost curve corresponds to p_g and the lower curves to p_α with $\alpha = 1, 0, -1/2$ in descending order.

Fig. 4: Steady-state interface width W_{SS}^2 versus system size L in dimension $3 + 1$, using distribution p_α with $\alpha = 1/2$. The fitted line has a slope of $2\chi = 0.48$. The inset shows the running exponent $\beta_{\text{eff}}(t)$ versus $t^{-\beta}$ with $\beta = 0.14$.

Fig. 5: Steady-state interface width W_{SS}^2 versus system size L in dimension $3 + 1$, using distribution p_α with $\alpha = -1/2$. The fitted line has a slope of $2\chi = 0.16$. The inset shows the running exponent $\beta_{\text{eff}}(t)$ versus $t^{-\beta}$ with $\beta = 0.05$.

Table. 1: Values of the measured growth exponent β as a function of dimension d , and noise distribution $p(\xi)$.

Table. 1

$d+1$	p_α					p_g
	$\alpha = -1/2$	$\alpha = 0$	$\alpha = 1/2$	$\alpha = 1$	$\alpha = 2$	
2+1	–	0.19(1)	–	–	–	0.24(1)
3+1	0.05(1)	0.11(1)	0.14(1)	0.15(1)	0.165(10)	0.185(10)
4+1	0.025(10)	0.07(1)	–	0.11(1)	–	0.14(1)

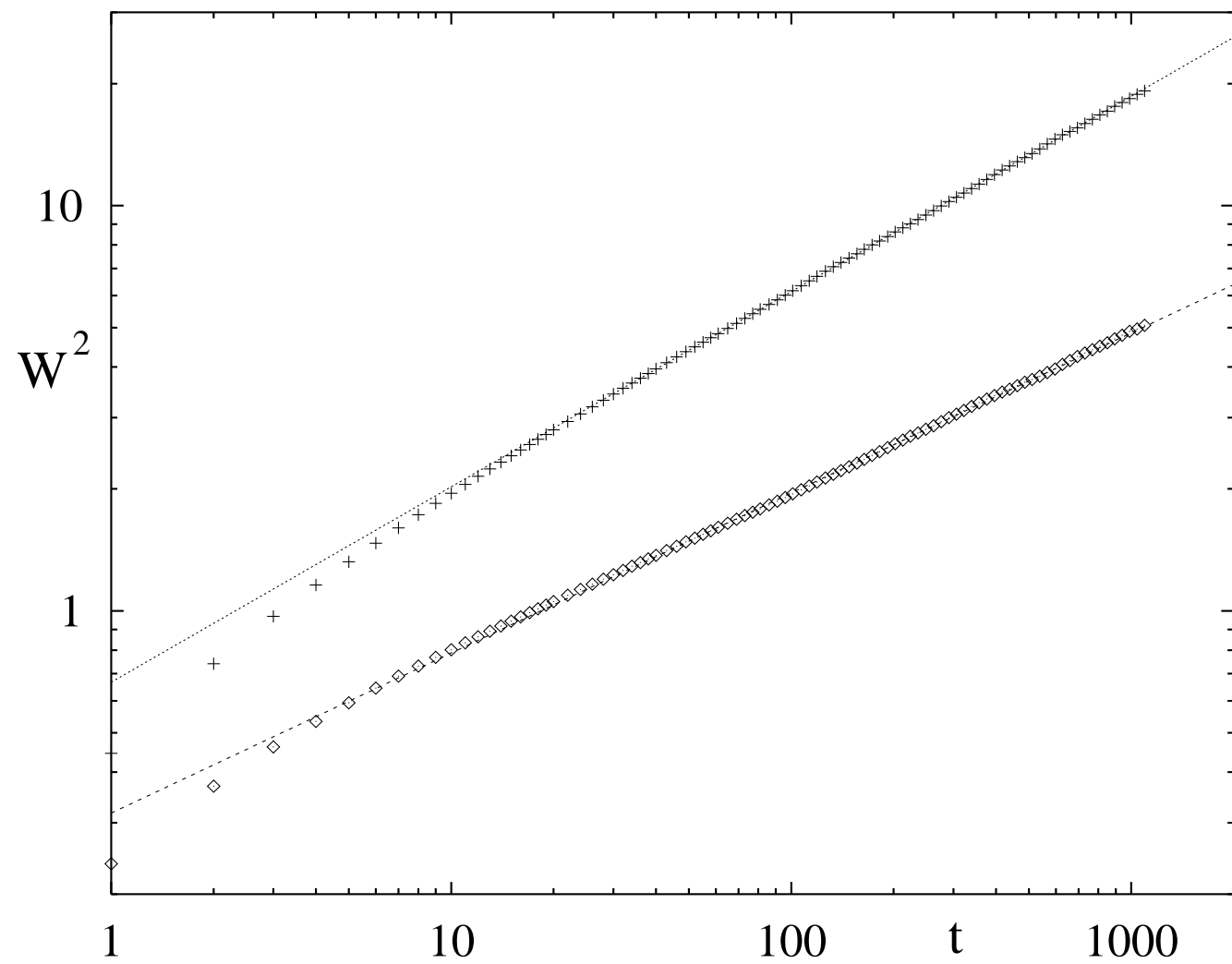


Figure 1, Newman and Swift, 'Non-universal exponents in interface growth'

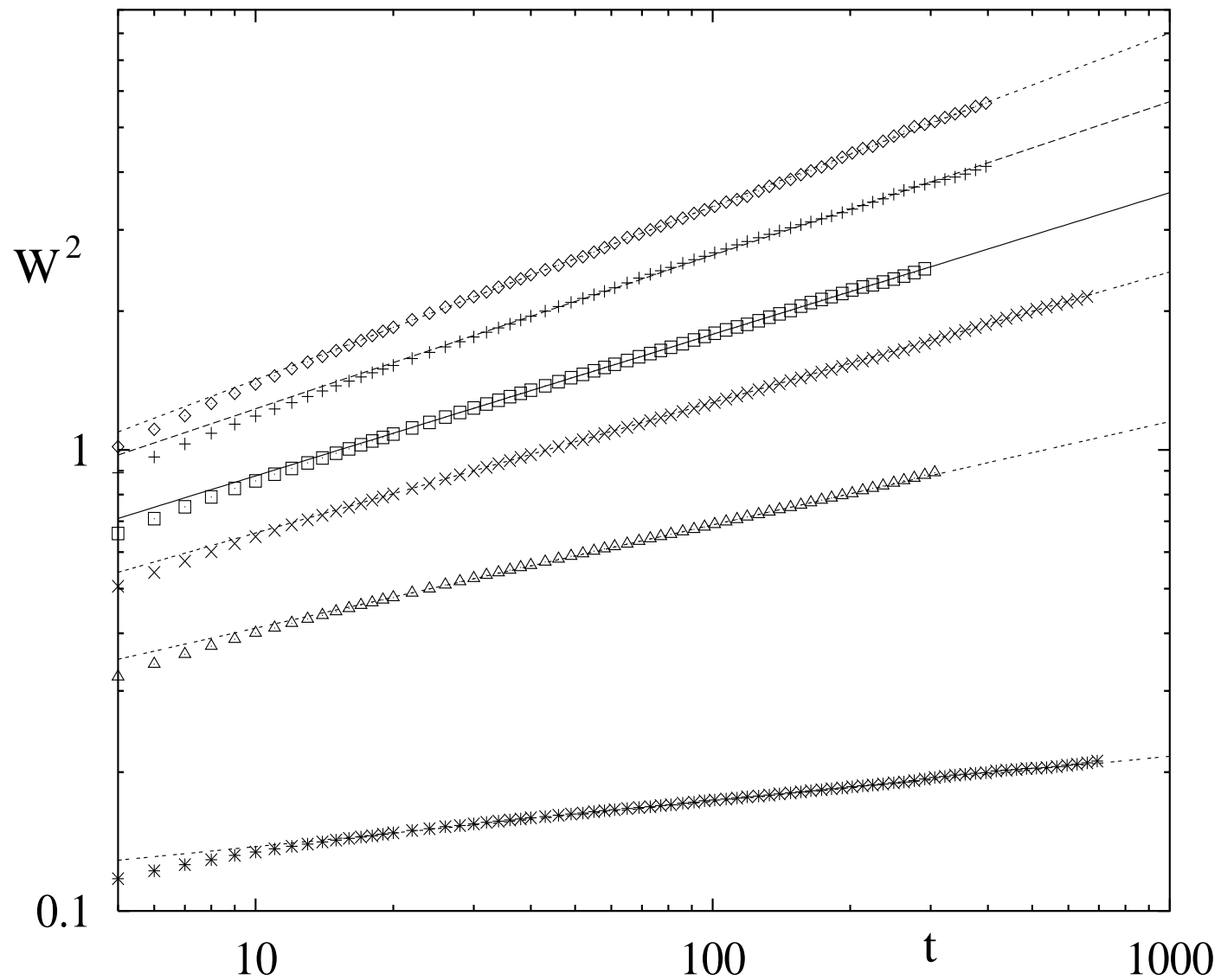


Figure 2, Newman and Swift, ‘Non-universal exponents in interface growth’

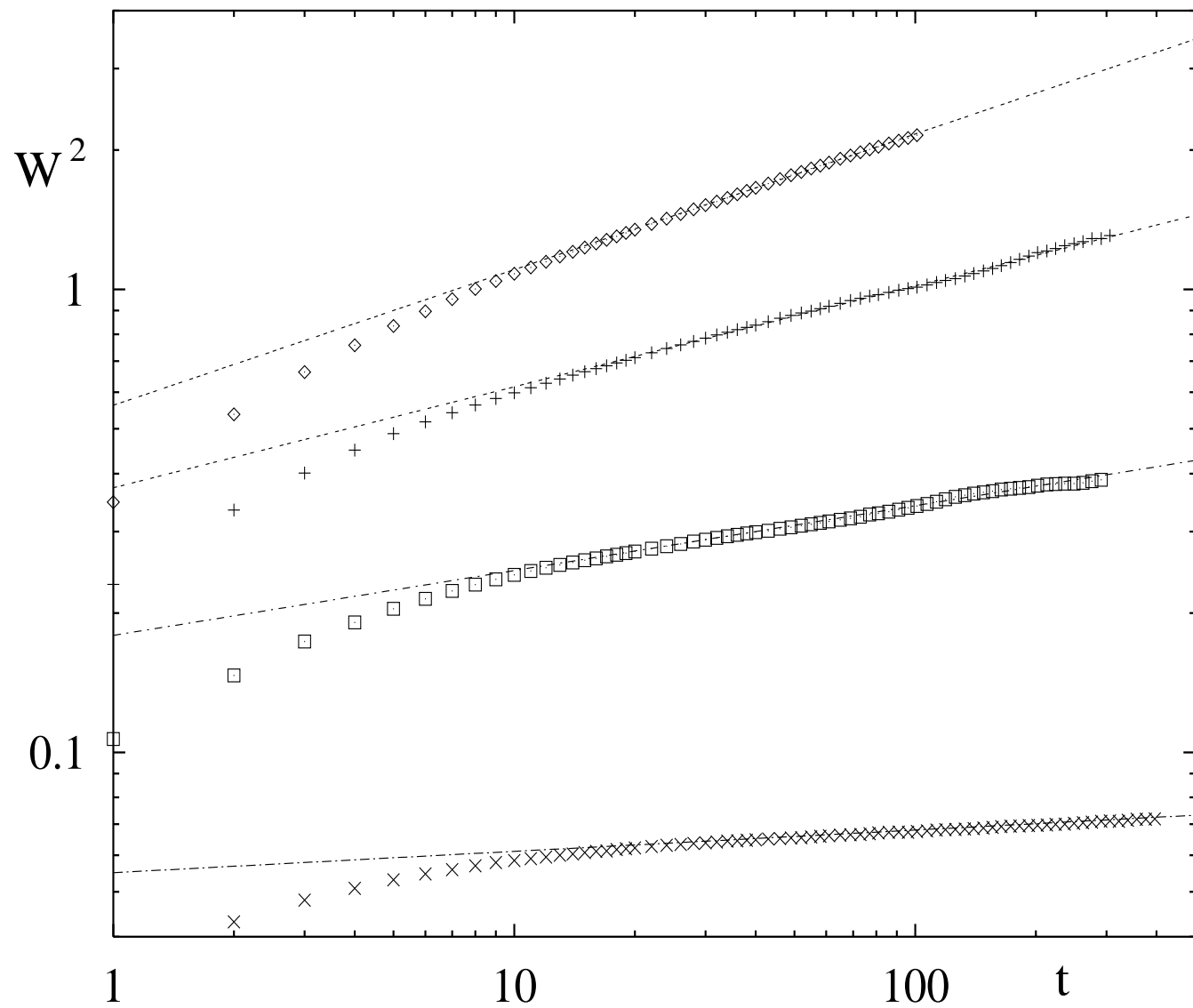


Figure 3, Newman and Swift, 'Non-universal exponents in interface growth'

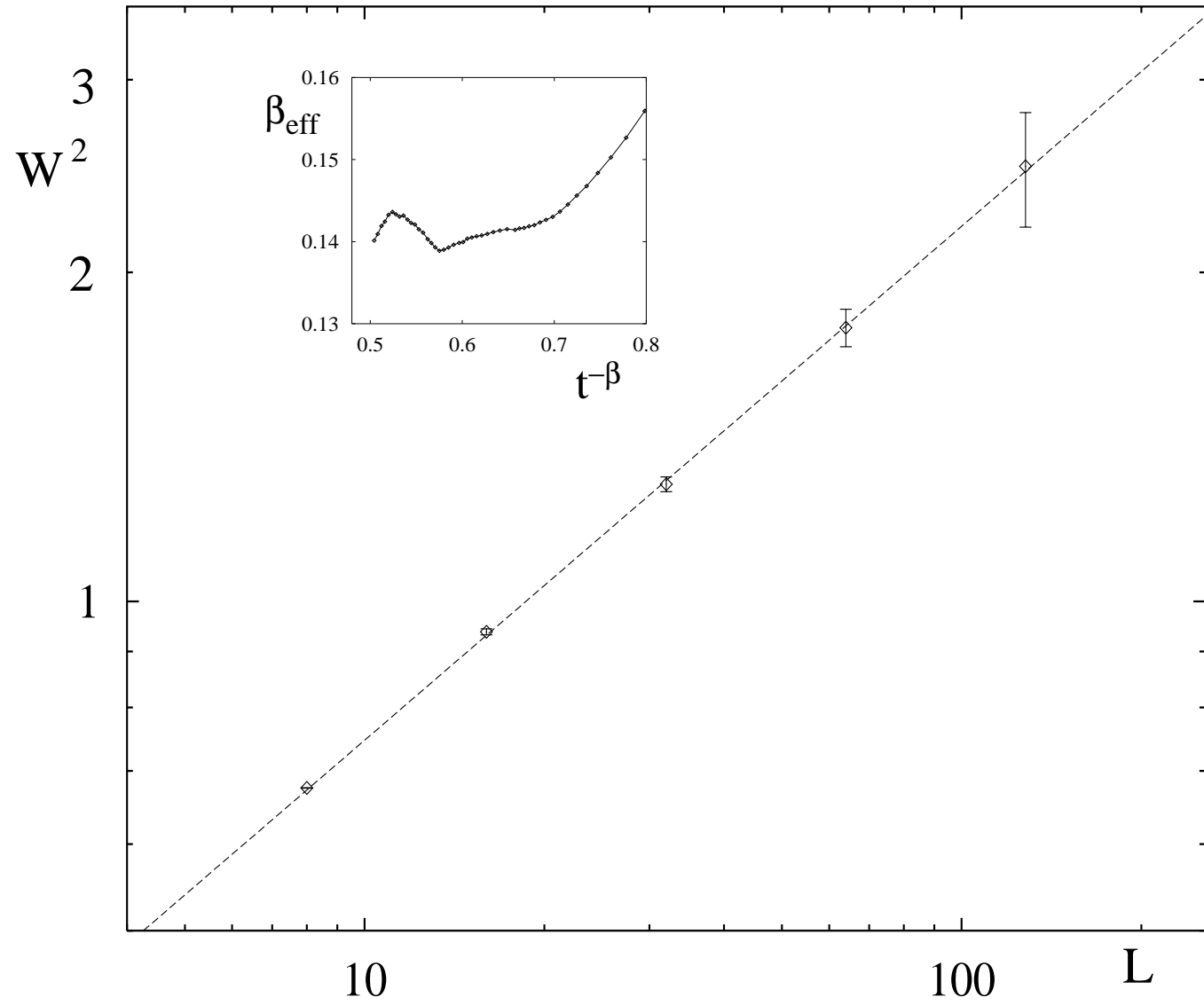


Figure 4, Newman and Swift, 'Non-universal exponents in interface growth'

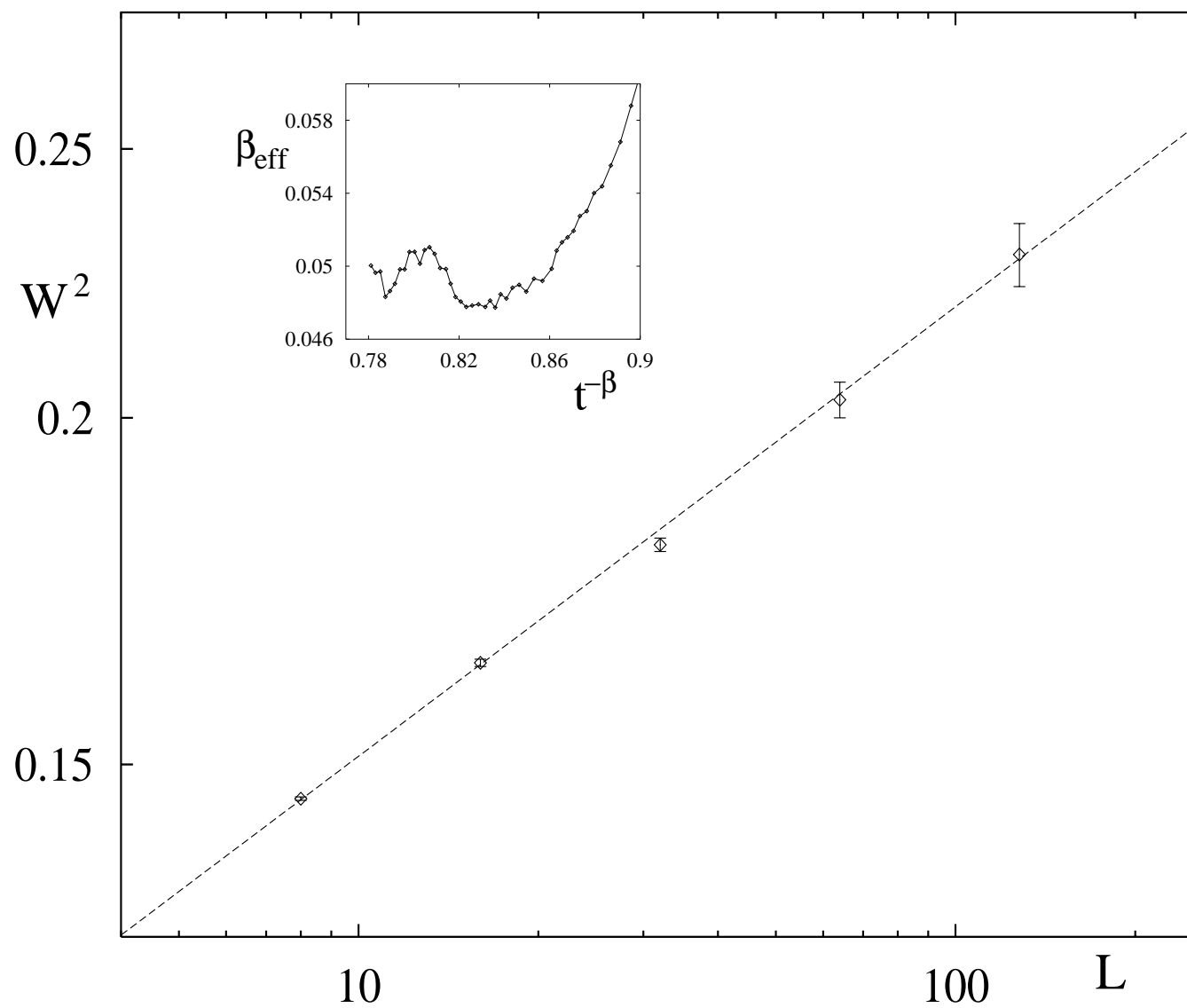


Figure 5, Newman and Swift, ‘Non-universal exponents in interface growth’